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Quasi-invariance properties of a class of subordinators

Max-K. von Renesse[a], Marc Yor[b], Lorenzo Zambotti[b,*]

[a] *Institut für Mathematik, TU Berlin, Strasse des 17. Juni 136, 10623 Berlin, Germany*
[b] *Laboratoire de Probabilités et Modèles Aléatoires (CNRS U.M.R. 7599), Université Paris 6 – Pierre et Marie Curie, U.F.R. Mathematiques, Case 188, 4 place Jussieu, 75252 Paris cedex 05, France*

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Abstract

We study absolute-continuity relationships for a class of stochastic processes, including the gamma and the Dirichlet processes. We prove that the laws of a general class of non-linear transformations of such processes are locally equivalent to the law of the original process and we compute explicitly the associated Radon–Nikodym densities. This work unifies and generalizes to random non-linear transformations several previous quasi-invariance results for gamma and Dirichlet processes.

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1. Introduction

In this paper we present several absolute-continuity results concerning, among others, the gamma process and the Dirichlet processes. We recall that the gamma process $(\gamma_t)_{t\geq 0}$ is a subordinator, i.e. a non-decreasing Lévy process, with gamma marginals, i.e. $\gamma_0 = 0$ and

$$\mathbb{P}(\gamma_t \in \mathrm{d}x) = p_t(x)\mathrm{d}x, \quad p_t(x) := 1_{[0,\infty)}(x)\frac{1}{\Gamma(t)}x^{t-1}\mathrm{e}^{-x}, \ t > 0, \ x \in \mathbb{R}.$$

Moreover for any $T > 0$, we define the Dirichlet process over $[0, T]$ as $D_t^{(T)} := \gamma_t/\gamma_T$, $t \in [0, T]$; we recall that γ_T is independent of $(\gamma_t/\gamma_T, t \in [0, T])$ and that, therefore,

* Corresponding author.
E-mail addresses: mrenesse@math.tu-berlin.de (M.-K. von Renesse), zambotti@ccr.jussieu.fr (L. Zambotti).

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$(D_t^{(T)}, t \in [0, T])$ is equal in law to the gamma process conditioned on $\{\gamma_T = 1\}$. See [14] for a survey of the main properties of the gamma process.

The gamma process has been the object of intensive research activity in recent years, both from pure and applied perspectives, such as in representation theory of infinite dimensional groups, in mathematical finance and in mathematical biology (see e.g. [12,3,5]). Quasi-invariance properties of the associated probability measure on path or measure space with respect to canonical transformations often play a central role. We recall that, given a measure μ on a space X and a measurable map $T : X \mapsto X$, quasi-invariance of μ under T means that μ and the image measure $T_*\mu$ are equivalent, i.e. mutually absolutely continuous. A classical example is the Girsanov formula for additive perturbations of Brownian motion (see, e.g., [9], Chap. VIII, and [11]).

In this paper we study quasi-invariance properties for a class of subordinators which we denote by $(\mathcal{L})$ and define below, with respect to a large class of non-linear sample path transformations. In particular, we unify and extend previous results on the real-valued gamma and Dirichlet processes.

Quasi-invariance properties of Lévy processes have been studied for quite some time; see e.g. Sato [10, p. 217–218]. In the case of the gamma process, for any measurable function $a : \mathbb{R}_+ \mapsto \mathbb{R}_+$ with a and $1/a$ bounded, the laws of $(\int_0^t a_s \mathrm{d}\gamma_s, t \geq 0)$ and $(\gamma_t, t \geq 0)$ are locally equivalent; see [12]. By local equivalence of two real-valued processes $(\eta_t, t \geq 0)$ and $(\zeta_t, t \geq 0)$, we mean that for all $T > 0$ the laws of $(\eta_t, t \in [0, T])$ and $(\zeta_t, t \in [0, T])$ are equivalent.

Here, we show the same property for a much wider class of path transformations of a $(\mathcal{L})$-subordinator ξ, e.g.

$$(\xi_t)_{t\geq 0} \mapsto \left(K(t, \xi_t) - \int_0^t \frac{\partial K}{\partial s}(s, \xi_s)\mathrm{d}s \right)_{t\geq 0} \quad \text{and} \quad (\xi_t)_{t\geq 0} \mapsto \left(\sum_{s\leq t} K(s, \Delta\xi_s) \right)_{t\geq 0},$$

where, for some $\alpha \in]0, 1[$, $K(s, \cdot)$ is a $C^{1,\alpha}$-isomorphism of $\mathbb{R}_+$ for each $s \geq 0$. We also establish analogous quasi-invariance results for transformations $D^{(T)} \mapsto K(D^{(T)})$ of the Dirichlet process, e.g.

$$(D_t^{(T)})_{t\in[0,T]} \mapsto (K(D_t^{(T)}))_{t\in[0,T]}, \quad \text{and} \quad (D_t^{(T)})_{t\in[0,T]} \mapsto \left(\sum_{s\leq t} H(s, \Delta D_s^{(T)}) \right)_{t\in[0,T]},$$

where $K(\cdot)$ and $H(s, \cdot)$ are increasing $C^{1,\alpha}$-homeomorphisms of $[0, 1]$ for each $s \in [0, T]$.

In all these cases, we compute the Radon–Nikodym density and study its martingale structure. We note that our approach allows us to treat the previously mentioned results of Vershik, Tsilevich and Yor [12,13], together with Handa's [5] and the recent work by von Renesse and Sturm [7] on Dirichlet processes, within a unified framework.

The paper ends with an application to SDEs driven by $(\mathcal{L})$-subordinators. Finally we point out that, in the same spirit as in [7], each quasi-invariance property we show yields easily an integration by parts formula on the path space; such formulae can be used in order to study an appropriate Dirichlet form and the associated infinite dimensional diffusion process. These applications will be developed in a future work.

1.1. The main result

Throughout our paper we fix a standard Borel space $(\Omega, \mathcal{F})$, in the sense of [6], endowed with a probability measure $\mathbb{P}$. All stochastic processes below will be defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

Let $(\xi_t)_{t\geq 0}$ be a subordinator, i.e. an increasing Lévy process with $\xi_0 = 0$. In this paper we consider subordinators in the class $(\mathcal{L})$, meaning *with logarithmic singularity*, i.e. we assume that ξ has zero drift and Lévy measure

$$\nu(\mathrm{d}x) = g(x)\mathrm{d}x, \quad x > 0,$$

where $g :]0, \infty[\mapsto \mathbb{R}_+$ is measurable and satisfies

(H1) $g > 0$ and $\int_1^\infty g(x)\mathrm{d}x < \infty$;
(H2) there exist $g_0 \geq 0$ and $\zeta : [0, 1] \mapsto \mathbb{R}$ measurable such that

$$g(x) = \frac{g_0}{x} + \zeta(x), \quad \forall x \in]0, 1], \quad \text{and} \quad \int_0^1 |\zeta(x)|\mathrm{d}x < +\infty.$$

We recall that for all $t \geq 0$, $\lambda > 0$

$$\mathbb{E}\left(\mathrm{e}^{-\lambda\xi_t}\right) = \exp(-t\,\Psi(\lambda)), \quad \Psi(\lambda) := \int_0^\infty \left(1 - \mathrm{e}^{-\lambda x}\right) g(x)\mathrm{d}x.$$

For the general theory of subordinators, see [4]. We denote by $\mathcal{F}_t := \sigma(\xi_s : s \leq t)$, $t \geq 0$, the filtration generated by ξ. We denote the space of càdlàg functions on $[0, t]$ by $\mathcal{D}([0, t])$, endowed with the Skorohod topology.

Remark 1.1. In the particular case of the gamma process $(\gamma_t)_{t\geq 0}$, mentioned above, we have

$$g(x) = \frac{\mathrm{e}^{-x}}{x}, \quad x > 0, \qquad \Psi(\lambda) = \log(1 + \lambda), \quad \lambda \geq 0.$$

Another remarkable example is the following:

$$g(x) = \frac{\mathrm{e}^{-ax}(1 - \mathrm{e}^{-bx})}{x(1 - \mathrm{e}^{-x})}, \quad x > 0,$$

where $a, b > 0$. If ξ is the associated subordinator, then $\mathrm{e}^{-\xi_1}$ is a Beta(a, b)-random variable: see [15]. For $a = 1$, the explicit value

$$\Psi(\lambda) = -\log\frac{\Gamma(1 + \lambda)\Gamma(1 + b)}{\Gamma(1 + \lambda + b)}$$

has been obtained in [8]. On the other hand, the stable subordinator of index $\alpha \in]0, 1[$ does not belong to the class $(\mathcal{L})$, since in this case the Lévy measure is $\nu(\mathrm{d}x) = Cx^{-1-\alpha}\mathrm{d}x$ on $\mathbb{R}_+$.

We consider a measurable function $h : \mathbb{R}_+ \times \Omega \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that

(1) h is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+)$-measurable, where $\mathcal{P}$ denotes the predictable σ-algebra generated by ξ;
(2) defining $h(s, a) = h(s, \omega, a)$, there exist finite constants $\kappa > 1$ and $\alpha \in]0, 1[$ such that almost surely

$$|h(s, x) - h(s, y)| \leq \kappa|x - y|^\alpha, \quad \forall x, y \in \mathbb{R}_+,\ s \geq 0, \tag{1.1}$$

$$0 < \kappa^{-1} \leq h(s, x) \leq \kappa < \infty, \quad \forall x \in \mathbb{R}_+,\ s \geq 0. \tag{1.2}$$

Then we set

$$H(s, x) = \int_0^x h(s, y)\mathrm{d}y, \quad \forall x \geq 0,\ s \geq 0.$$

Note that a.s. $H(s,\cdot):\mathbb{R}_+\mapsto\mathbb{R}_+$ is necessarily a C^1-diffeomorphism for all $s\geq 0$. We set

$$\Delta\xi_s := \xi_s - \xi_{s-}, \quad s\geq 0,$$

and we denote by $\mathcal{J}_t^\xi$ the set of jump times of ξ up to time $t\geq 0$:

$$\mathcal{J}_t^\xi := \{s\in[0,t] : \Delta\xi_s \neq 0\}. \tag{1.3}$$

We can now state the main result of this paper:

Theorem 1.2. (1) *The process* M_t^H,

$$M_t^H := \exp\left(g_0\int_0^t \log h(s,0)\mathrm{d}s\right)\prod_{s\in\mathcal{J}_t^\xi}\left[h(s,\Delta\xi_s)\cdot\frac{g(H(s,\Delta\xi_s))}{g(\Delta\xi_s)}\right], \quad t\geq 0,$$

is an $(\mathcal{F}_t,\mathbb{P})$*-martingale with* $\mathbb{E}(M_t^H)=1$ *and a.s.* $M_t^H>0$. *We can uniquely define a probability measure* $\mathbb{P}^H$ *such that* $\mathbb{P}^H_{|\mathcal{F}_t} = M_t^H\cdot\mathbb{P}_{|\mathcal{F}_t}$ *for all* $t\geq 0$.

(2) *Setting*

$$\xi_t^H := \sum_{s\leq t} H(s,\Delta\xi_s), \quad t\geq 0, \tag{1.4}$$

then ξ^H *is distributed under* $\mathbb{P}^H$ *as* ξ *under* $\mathbb{P}$.

Note that Theorem 1.2 presents a local equivalence result for the laws of ξ and ξ^H, since a.s. $M_t^H>0$. The theorem is stated for general subordinators in the class $(\mathcal{L})$ defined above and for a general random transformation; in Section 3 we consider some special cases of the general result, and in Section 4 we consider the case of the Dirichlet process.

Once the equivalence result of Theorem 1.2 is proven, it is interesting to study the associated Radon–Nikodym density. Note that the density M_t^H is by construction $\sigma(\xi_s, s\in[0,t])$-measurable for all $t\geq 0$.

We denote by $X_t:\mathcal{D}([0,+\infty))\mapsto\mathbb{R}$ the coordinate process: $X_t(w)=w_t$, $t\geq 0$, and by $\mathcal{G}_t=\sigma(X_s, s\in[0,t])$ the natural filtration of X. Our main purpose in stating the following lemma is to fix notation for important quantities in the sequel of this paper.

Lemma 1.3. (1) *For all* $t\in\mathbb{R}_+$ *let*

$$N_t := \mathbb{E}\left(M_t^H|\sigma(\xi_s^H, s\in[0,t])\right) = \rho_t(\xi_s^H, s\in[0,t]) \quad a.s.,$$

where $\rho_t:\mathcal{D}([0,t])\mapsto\mathbb{R}_+$ *is some Borel functional. Then the Radon–Nikodym density of the law of* $(\xi_s^H, s\in[0,t])$ *with respect to the law of* $(\xi_s, s\in[0,t])$ *is* $1/\rho_t$, *i.e.*

$$\mathbb{P}(\xi^H\in A)=\mathbb{E}\left(1_A(\xi)\frac{1}{\rho_t(\xi)}\right), \quad \forall A\in\mathcal{G}_t. \tag{1.5}$$

(2) *If, for all* $t\geq 0$, $(\xi_s, s\in[0,t])$ *is* $\sigma(\xi_s^H, s\in[0,t])$*-measurable, i.e.* $(\xi_s, s\in[0,t]) = F_t(\xi_s^H, s\in[0,t])$ *a.s. for some measurable* $F_t:\mathcal{D}([0,t])\mapsto\mathcal{D}([0,t])$, *then*

$$N_t = M_t^H(\xi_s, s\in[0,t]) = M_t^H(F_t(\xi_s^H, s\in[0,t]))$$

and Eq. (1.5) *can be rewritten as*

$$\mathbb{P}(\xi^H\in A)=\mathbb{E}\left(1_A(\xi)\frac{1}{M_t(F_t(\xi_\cdot))}\right), \quad \forall A\in\mathcal{G}_t. \tag{1.6}$$

We remark that $(\xi^H_s, s \in [0,t])$ is $\sigma(\xi_s, s \in [0,t])$-measurable for all $t \geq 0$ by the definition of ξ^H. The converse statement (2) in Lemma 1.3 can be proven in several cases of interest, but might not be true in the most general setting of Theorem 1.2. An explicit computation of the Radon–Nikodym density of the law of ξ^H with respect to the law of ξ depends on the explicit computation of the functionals ρ_t or F_t defined in Lemma 1.3.

1.2. A parallel between the gamma process and Brownian motion

The absolute-continuity results presented in this paper can be better understood by comparison with some analogous properties of Brownian motion.

The Girsanov theorem for a Brownian motion $(B_t, t \geq 0)$ states the following property: if $(a_s, s \geq 0)$ is an adapted and (say) bounded process, then the law of the process

$$t \mapsto B_t + \int_0^t a_s \mathrm{d}s, \quad t \geq 0,$$

is locally equivalent to that of $(B_t, t \geq 0)$, with explicit Radon–Nikodym density. We call this property quasi-invariance by addition.

As a by-product case of our Theorem 1.2, the gamma process γ has an analogous property of quasi-invariance by multiplication (see also [12]): if $(a_s, s \geq 0)$ is a predictable process such that a and $1/a$ are bounded, then the law of

$$t \mapsto \int_0^t a_s \mathrm{d}\gamma_s, \quad t \geq 0,$$

is locally equivalent to that of $(\gamma_t, t \geq 0)$, and we compute explicitly the Radon–Nikodym density. In fact, we can prove the same quasi-invariance property for all $(\mathcal{L})$-subordinators.

The Girsanov theorem for Brownian motion has important applications in the study of stochastic differential equations (SDEs) driven by a Wiener process; likewise, our Theorem 1.2 allows us to give analogous applications to SDEs driven by $(\mathcal{L})$-subordinators, e.g. to compute explicitly laws of solutions; see Section 5.

1.3. A look at the bibliography

In the following sections we prove Theorem 1.2 and we give applications to quasi-invariance properties of $(\mathcal{L})$-subordinators, the γ process and the Dirichlet process.

Some particular examples of our applications have already been studied in the literature: necessary and sufficient conditions for two subordinators to have equivalent laws are given in Sato [10, p. 217–218]; we have already mentioned the result of Tsilevich, Vershik and Yor in [12] concerning local equivalence of γ and $(\int_0^t a_s \mathrm{d}\gamma_s, t \geq 0)$, for any deterministic measurable function $a : \mathbb{R}_+ \mapsto \mathbb{R}_+$ with a and $1/a$ bounded, with explicit Radon–Nikodym density; in the case of the Dirichlet process, two distinct quasi-invariance properties have been studied by Handa in [5] and von Renesse and Sturm in [7]. We refer to the remarks after each result in Sections 3 and 4, where these results are recalled in detail.

2. A generalization of a formula of Tsilevich, Vershik and Yor

Within the framework of Section 1.1, the law of ξ with $\xi_0 = 0$ is characterized by its Laplace transform, i.e. for any measurable bounded $\lambda : \mathbb{R}_+ \mapsto \mathbb{R}_+$

$$\mathbb{E}\left[\exp\left(-\int_0^t \lambda_s \mathrm{d}\xi_s\right)\right] = \exp\left(-\int_0^t \Psi(\lambda_s)\mathrm{d}s\right).$$

In order to prove Theorem 1.2, we shall show that ξ^H has, under $\mathbb{P}^H$, the same Laplace transform as ξ under $\mathbb{P}$, namely for all measurable bounded $\lambda : \mathbb{R}_+ \mapsto \mathbb{R}_+$

$$\mathbb{E}^H \left[\exp \left(- \int_0^t \lambda_s \mathrm{d}\xi_s^H \right) \right] = \exp \left(- \int_0^t \Psi(\lambda_s) \mathrm{d}s \right).$$

To do that, we shall show that the process

$$\exp \left(- \int_0^t \lambda_s \mathrm{d}\xi_s^H + \int_0^t \Psi(\lambda_s) \mathrm{d}s \right), \quad t \geq 0,$$

is an $((\mathcal{F}_t), \mathbb{P}^H)$-martingale, which is equivalent to proving the following

Proposition 2.1. *We set for all $t \geq 0$*

$$M_t^{H,\lambda} := \exp \left(\int_0^t (g_0 \log h(s,0) + \Psi(\lambda_s)) \, \mathrm{d}s \right) \cdot \\ \cdot \prod_{s \in \mathcal{J}_t^\xi} \left[h(s, \Delta\xi_s) \frac{g(H(s, \Delta\xi_s))}{g(\Delta\xi_s)} \exp(-\lambda_s H(s, \Delta\xi_s)) \right]. \tag{2.1}$$

Then $M^{H,\lambda}$ is a $(\mathcal{F}_t, \mathbb{P})$-martingale with $\mathbb{E}(M_t^{H,\lambda}) = 1$ and a.s. $M_t^{H,\lambda} > 0$.

Tsilevich, Vershik and Yor prove in [12] the same result for ξ a gamma process and $H(s,x) = c(s)x$, for $c : \mathbb{R}_+ \mapsto \mathbb{R}_+$ measurable and deterministic.

We say that a real-valued process $(\zeta_t, t \geq 0)$ has bounded variation if a.s. for all $T > 0$ the real-valued function $[0, T] \ni t \mapsto \zeta_t$ has bounded variation.

Lemma 2.2. *Let us have $F : \mathbb{R}_+ \times \Omega \times \mathbb{R}_+ \mapsto]-1, \infty[$ such that*

- *F is $\mathcal{P} \otimes \mathcal{B}_{\mathbb{R}_+}$-measurable, where $\mathcal{P}$ denotes the predictable σ-algebra generated by ξ;*
- *there exists a finite constant C_F such that a.s. for almost every $s \geq 0$*

$$F(s,0) = 0, \quad \int \nu(\mathrm{d}x) \mathbb{E}[|F(s,x)|] \leq C_F < \infty. \tag{2.2}$$

Then

(1) *the process*

$$x_t^F := \sum_{s \leq t} F(s, \Delta\xi_s) - \int_0^t \mathrm{d}s \int \nu(\mathrm{d}x) F(s,x), \quad t \geq 0$$

is a martingale with bounded variation;

(2) *the process*

$$\mathcal{E}_t^F := \exp \left(- \int_0^t \mathrm{d}s \int \nu(\mathrm{d}x) F(s,x) \right) \prod_{s \leq t} (1 + F(s, \Delta\xi_s))$$

satisfies

$$\mathcal{E}_t^F = 1 + \int_0^t \mathcal{E}_{s-}^F \mathrm{d}x_s^F, \quad t \geq 0; \tag{2.3}$$

moreover $(\mathcal{E}_t^F)$ *is a martingale with bounded variation which satisfies*

$$\mathbb{E}\left(\int_0^t \left|\mathrm{d}\mathcal{E}_u^F\right|\right) \le 2C_F t;$$

(3) *for all* $t \ge 0$*, a.s.* $\mathcal{E}_t^F > 0$.

Proof. Note first that x^F is well defined, since by Eq. (2.2)

$$\mathbb{E}\left[\sum_{s\le t} |F(s,\Delta\xi_s)|\right] = \int_0^t \mathrm{d}s \int \nu(\mathrm{d}x)\mathbb{E}\left[|F(s,x)|\right] \le C_F t < \infty.$$

Since $\{(s,\Delta\xi_s), s \ge 0\}$ is a Poisson point process with intensity measure $\mathrm{d}s\,\nu(\mathrm{d}x)$, it follows immediately that x^F is a local martingale. Furthermore, a.s. the paths of x^F have bounded variation, since

$$\mathbb{E}\left(\int_{(s,t]} \left|\mathrm{d}x_u^F\right| \,\middle|\, \mathcal{F}_s\right) \le 2C_F(t-s), \quad t > s \ge 0. \tag{2.4}$$

Therefore, $(x_t^F, t \ge 0)$ is a true martingale; indeed, for any $t > 0$, $\sup_{s\le t}|x_s^F| \le \int_0^t \left|\mathrm{d}x_s^F\right|$, and therefore

$$\mathbb{E}\left(\sup_{s\le t}|x_s^F|\right) \le 2C_F t, \quad t \ge 0;$$

by Proposition IV.1.7 of [9] we obtain the claim.

Since $\mathcal{E}^F$ is the Doléans exponential associated with the martingale x^F, i.e. it satisfies Eq. (2.3), it is clear that $\mathcal{E}^F$ is a local martingale (see chapter 5 of [1]). Moreover, since $\mathcal{E}^F$ is non-negative, then it is a super-martingale and in particular $\mathbb{E}\left(\mathcal{E}_t^F\right) \le \mathbb{E}\left(\mathcal{E}_0^F\right) = 1$. Furthermore, by Eq. (2.4)

$$\mathbb{E}\left(\int_0^t \left|\mathrm{d}\mathcal{E}_u^F\right|\right) = \mathbb{E}\left(\int_0^t \mathcal{E}_{u-}^F \left|\mathrm{d}x_u^F\right|\right) \le \int_0^t \mathbb{E}\left(\mathcal{E}_u^F\right) 2C_F \mathrm{d}u \le 2C_F t.$$

The same argument as for x^F yields

$$\mathbb{E}\left(\sup_{s\le t}\left|\mathcal{E}_s^F\right|\right) \le 2C_F t, \quad t \ge 0,$$

and therefore $\mathcal{E}^F$ is a martingale.

In order to prove that $\mathcal{E}_t^F > 0$ a.s., by Eq. (2.2) it is enough to show that

$$\log \prod_{s\le t}(1+F(s,\Delta\xi_s)) = \sum_{s\le t}\log(1+F(s,\Delta\xi_s)) > -\infty.$$

Since $F(s,\Delta\xi_s) = \Delta x_s^F = x_s^F - x_{s-}^F > -1$, and x^F has a.s. bounded variation, then there are a.s. only a finite number of $s \in [0,t]$ such that $\Delta x_s^F < -1/2$ and therefore a.s. $\inf_{s\le t}\Delta x_s^F =: C_t > -1$. It follows that

$$\sum_{s\le t}\log\left(1+\Delta x_s^F\right) \ge -\frac{1}{C_t+1}\sum_{s\le t}\left|\Delta x_s^F\right| = -\frac{1}{C_t+1}\int_0^t\left|\mathrm{d}x_u^F\right| > -\infty, \quad \text{a.s.} \quad \square$$

The main steps in the proofs of Proposition 2.1 and Theorem 1.2 are the estimate Eq. (2.6) and the identity Eq. (2.7) below, which allow us to apply Lemma 2.2 to

$$F(s,0) := 0, \qquad F(s,x) := h(s,x)\cdot \frac{g(H(s,x))}{g(x)}\cdot \mathrm{e}^{-\lambda_s H(s,x)} - 1, \quad x > 0. \tag{2.5}$$

These two important points are gathered in the following:

Lemma 2.3. *Let $\phi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ a C^1 function such that $\phi(0) = 0$,*

$$0 < \kappa^{-1} \le \phi'(x) \le \kappa < \infty, \quad |\phi'(x) - \phi'(y)| \le \kappa|x-y|^\alpha, \quad \forall x, y \in \mathbb{R}_+,$$

where $\kappa > 1$ and $\alpha \in]0, 1[$. We set for all $a \ge 0$

$$F_{a,\phi} :]0,\infty[\mapsto \mathbb{R}, \quad F_{a,\phi} := \phi' \cdot \frac{g(\phi)}{g}\cdot \mathrm{e}^{-a\phi} - 1.$$

Then

$$\int_0^\infty \left|F_{a,\phi}(x)\right| g(x)\mathrm{d}x \le C(\kappa,\alpha,a), \tag{2.6}$$

and

$$\int_0^\infty F_{a,\phi}(x) g(x)\mathrm{d}x = -\Psi(a) - g_0 \log\phi'(0), \tag{2.7}$$

where

$$C(\kappa,\alpha,a) = a g_0 + \frac{g_0\kappa^2}{\alpha(1+\alpha)} + 2\int_{\kappa^{-2}}^\infty g\mathrm{d}x + 3\int_0^1 |\zeta|\mathrm{d}x < +\infty.$$

Remark 2.4. Since formula (2.7) is crucial in our discussion, we would like to give some intuition about it. By a formal differentiation we find

$$\frac{\partial}{\partial a}\int_0^\infty F_{a,\phi}(x)g(x)\mathrm{d}x = -\int_0^\infty \phi(x)\mathrm{e}^{-a\phi(x)} g(\phi(x))\phi'(x)\mathrm{d}x.$$

With the change of variable $y = \phi(x)$ we obtain from the last expression

$$\frac{\partial}{\partial a}\int_0^\infty F_{a,\phi}(x)g(x)\mathrm{d}x = -\int_0^\infty y\mathrm{e}^{-ay} g(y)\mathrm{d}y = -\Psi'(a).$$

Therefore, the function $a \mapsto \int_0^\infty F_{a,\phi}(x)g(x)\mathrm{d}x + \Psi(a)$ is constant. In fact, Eq. (2.7) shows that this constant only depends on $\phi'(0)$.

Proof of Lemma 2.3. Notice that $\phi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a diffeomorphism. First we have

$$\begin{aligned}\int_{\kappa^{-1}}^\infty \left|F_{a,\phi}\right| g\mathrm{d}x &\le \int_{\kappa^{-1}}^\infty \phi' g(\phi)\mathrm{d}x + \int_{\kappa^{-1}}^\infty g\mathrm{d}x \\ &= \int_{\phi(\kappa^{-1})}^\infty g(y)\mathrm{d}y + \int_{\kappa^{-1}}^\infty g\mathrm{d}x \le 2\int_{\kappa^{-2}}^\infty g(x)\mathrm{d}x < \infty.\end{aligned}$$

Now

$$\int_0^{\kappa^{-1}} \left|F_{a,\phi}\right| g\mathrm{d}x = \int_0^{\kappa^{-1}} \left|\phi' g(\phi)\mathrm{e}^{-a\phi} - g\right| \mathrm{d}x$$

$$
\begin{aligned}
&\le \int_0^{\kappa^{-1}} \phi' g(\phi)\left(1-\mathrm{e}^{-a\phi}\right)\mathrm{d}x + \int_0^{\kappa^{-1}} \phi' \left|g(\phi) - \frac{g_0}{\phi}\right| \mathrm{d}x \\
&\quad + \int_0^1 g_0 \left|\frac{\phi'(x)}{\phi(x)} - \frac{1}{x}\right| \mathrm{d}x \\
&\quad + \int_0^{\kappa^{-1}} \left|\frac{g_0}{x} - g(x)\right| \mathrm{d}x =: I_0 + I_1 + I_2 + I_3.
\end{aligned}
$$

First we estimate I_2.

$$
\begin{aligned}
I_2 &= \int_0^1 g_0 \left|\frac{\phi'(x)}{\phi(x)} - \frac{1}{x}\right| \mathrm{d}x = g_0 \int_0^1 \left|\frac{\phi(x) - x\phi'(x)}{x\phi(x)}\right| \mathrm{d}x \\
&\le \int_0^1 \frac{g_0}{\kappa^{-1}x^2} \left|\int_0^x \left[\phi'(y) - \phi'(x)\right]\mathrm{d}y\right| \mathrm{d}x \le g_0\kappa^2 \int_0^1 \frac{1}{x^2}\int_0^x y^\alpha \mathrm{d}y\mathrm{d}x = \frac{g_0\kappa^2}{\alpha(1+\alpha)}.
\end{aligned}
$$

Recall now that $g(x) = \frac{g_0}{x} + \zeta(x)$ by (H2) above. Then I_3 can be estimated by

$$
I_3 = \int_0^{\kappa^{-1}} \left|\frac{g_0}{x} - g(x)\right| \mathrm{d}x \le \int_0^1 |\zeta| \mathrm{d}x.
$$

Then I_0 and I_1 can be estimated similarly by changing variable:

$$
I_1 = \int_0^{\kappa^{-1}} \phi' \left|g(\phi) - \frac{g_0}{\phi}\right| \mathrm{d}x = \int_0^{\phi(\kappa^{-1})} \left|g(x) - \frac{g_0}{x}\right| \mathrm{d}x \le \int_0^1 |\zeta| \mathrm{d}x,
$$

and

$$
I_0 = \int_0^{\kappa^{-1}} \phi' g(\phi)\left(1-\mathrm{e}^{-a\phi}\right)\mathrm{d}x = \int_0^{\phi(\kappa^{-1})} g(x)(1-\mathrm{e}^{-ax})\mathrm{d}x \le a g_0 + \int_0^1 |\zeta| \mathrm{d}x,
$$

since $\phi(\kappa^{-1}) \le 1$. Therefore, we have obtained

$$
\int_0^\infty \left|F_{a,\phi}\right| g \mathrm{d}x \le a g_0 + \frac{g_0\kappa^2}{\alpha(1+\alpha)} + 2\int_{\kappa^{-2}}^\infty g(y)\mathrm{d}y + 3\int_0^1 |\zeta(x)|\mathrm{d}x,
$$

and Eq. (2.6) is proven.

We turn now to the proof of Eq. (2.7). By Eq. (2.6) and dominated convergence

$$
\int_0^\infty F_{a,\phi} g \mathrm{d}x = \lim_{\varepsilon \searrow 0} \int_\varepsilon^\infty F_{a,\phi} g \mathrm{d}x.
$$

For all $\varepsilon > 0$ we have by the change of variable $y = \phi(x)$

$$
\int_\varepsilon^\infty \phi' g(\phi)\mathrm{e}^{-a\phi}\mathrm{d}x = \int_{\phi(\varepsilon)}^\infty g(y)\mathrm{e}^{-ay}\mathrm{d}y.
$$

Then we want to compute the limit as $\varepsilon \searrow 0$ of

$$
\begin{aligned}
\int_\varepsilon^\infty F_{a,\phi} g \mathrm{d}x &= \int_{\phi(\varepsilon)}^\infty g(x)\mathrm{e}^{-ax}\mathrm{d}x - \int_\varepsilon^\infty g(x)\mathrm{d}x \\
&= \int_{\phi(\varepsilon)}^\infty g(x)\left(\mathrm{e}^{-ax} - 1\right)\mathrm{d}x + \int_{\phi(\varepsilon)}^1 g(x)\mathrm{d}x - \int_\varepsilon^1 g(x)\mathrm{d}x.
\end{aligned}
$$

Clearly, by assumptions (H1)–(H2) and by dominated convergence

$$\lim_{\varepsilon \searrow 0} \int_{\phi(\varepsilon)}^{\infty} g(x) \left(\mathrm{e}^{-ax} - 1\right) \mathrm{d}x = \int_0^{\infty} g(x) \left(\mathrm{e}^{-ax} - 1\right) \mathrm{d}x = -\Psi(a).$$

Now, by assumption (H2)

$$\begin{aligned} \lim_{\varepsilon \searrow 0} \left[\int_{\phi(\varepsilon)}^{1} g(x)\mathrm{d}x - \int_{\varepsilon}^{1} g(x)\mathrm{d}x \right] &= g_0 \lim_{\varepsilon \searrow 0} \left[\int_{\phi(\varepsilon)}^{1} \frac{1}{x}\mathrm{d}x - \int_{\varepsilon}^{1} \frac{1}{x}\mathrm{d}x \right] \\ &= g_0 \lim_{\varepsilon \searrow 0} \log \frac{\varepsilon}{\phi(\varepsilon)} = -g_0 \log \phi'(0). \end{aligned}$$

Then we have obtained Eq. (2.7). □

Proof of Proposition 2.1. It is enough to apply the results of Lemmas 2.2 and 2.3 to $\phi(x) := H(s,x)$, $a = \lambda_s$ and F defined in (2.5). Positivity of $\mathcal{E}_t^F = M_t^{H,\lambda}$ follows from point (3) of Lemma 2.2. □

Proof of Theorem 1.2. Notice that $M^H = M^{H,\lambda}$ for $\lambda \equiv 0$. By Proposition 2.1, M^H is a martingale with expectation 1. Then, for any bounded measurable $\lambda : \mathbb{R}_+ \mapsto \mathbb{R}_+$, by Proposition 2.1 we obtain

$$\mathbb{E}\left(\exp\left(-\int_0^t \lambda_s \mathrm{d}\xi_s^H\right) M_t^H\right) = \exp\left(-\int_0^t \Psi(\lambda_s)\mathrm{d}s\right), \quad t \geq 0.$$

The desired result now follows by uniqueness of the Laplace transform. □

3. Quasi-invariance properties of ($\mathcal{L}$)-subordinators

In this section we point out two special cases of Theorem 1.2. Throughout the paper we consider a measurable function $k : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ which satisfies, for some finite constants $\kappa \geq 1$ and $\alpha \in]0, 1[$,

$$|k(s,x) - k(s,y)| \leq \kappa |x - y|^{\alpha}, \quad \forall s, x, y \in \mathbb{R}_+, \tag{3.1}$$

$$0 < \kappa^{-1} \leq k(s,x) \leq \kappa < \infty, \quad \forall s, x \in \mathbb{R}_+, \tag{3.2}$$

and we set

$$K(s,x) := \int_0^x k(s,y)\mathrm{d}y, \quad \forall x, s \geq 0. \tag{3.3}$$

Notice that $K(s,\cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is necessarily bijective for all $s \geq 0$, so that there exists an inverse

$$R(s,\cdot) : \mathbb{R}_+ \mapsto \mathbb{R}_+, \qquad K(s, R(s,x)) = x, \quad \forall\, x \in \mathbb{R}_+. \tag{3.4}$$

Notice that R satisfies analogs of Eqs. (3.1)–(3.3). For any increasing function $w : \mathbb{R}_+ \mapsto \mathbb{R}_+$ we define

$$w_t^K := \sum_{s \leq t} K(s, w_s - w_{s-}), \quad t \geq 0. \tag{3.5}$$

Note that

$$\left(w^K\right)_t^R := w_t, \quad t \geq 0. \tag{3.6}$$

3.1. Quasi-invariance of ξ under composition with a diffeomorphism

In this subsection we also assume

$$\text{for all } x \in \mathbb{R}_+, \quad K(\cdot, x) \in C^1(\mathbb{R}_+). \tag{3.7}$$

Then for any increasing function $w : \mathbb{R}_+ \mapsto \mathbb{R}_+$ we define

$$(K \circ w)_t := K(t, w_t) - \int_0^t \frac{\partial K}{\partial s}(s, w_s)\mathrm{d}s, \quad t \geq 0. \tag{3.8}$$

We set $H : \mathbb{R}_+ \times \Omega \times \mathbb{R}_+ \mapsto [0, \infty[$,

$$H(s, x) := K(s, \xi_{s-} + x) - K(s, \xi_{s-}), \qquad h(s, x) := k(s, \xi_{s-} + x), \quad s, x \geq 0,$$

and we notice that H is $\mathcal{P} \otimes \mathcal{B}_{\mathbb{R}_+}$-measurable. By the chain rule (Proposition 0.4.6 in [9])

$$K(t, \xi_t) = \int_0^t \frac{\partial K}{\partial s}(s, \xi_s)\mathrm{d}s + \sum_{s \leq t} \left[K(s, \xi_s) - K(s, \xi_{s-})\right], \quad \forall t \geq 0.$$

Since $K(s, \xi_s) - K(s, \xi_{s-}) = H(s, \Delta\xi_s)$ for all $s \geq 0$, we obtain that

$$(K \circ \xi)_t = \sum_{s \leq t} H(s, \Delta\xi_s), \quad t \geq 0.$$

Moreover Eqs. (1.1) and (1.2) are satisfied and Theorem 1.2 and Lemma 1.3 yield

Corollary 3.1. (1) *The process*

$$G_t^K(\xi) := \exp\left(g_0 \int_0^t \log k(s, \xi_s)\mathrm{d}s\right) \prod_{s \in \mathcal{J}_t^\xi} \left[k(s, \xi_s) \cdot \frac{g(K(s, \xi_s) - K(s, \xi_{s-}))}{g(\Delta\xi_s)}\right],$$
$$t \geq 0,$$

is a non-negative $(\mathcal{F}_t)$-martingale with $\mathbb{E}(G_t^K(\xi)) = 1$ and a.s. $G_t^K(\xi) > 0$.

(2) *Let $\mathbb{P}^K$ be the unique probability measure on $(\Omega, \mathcal{F})$ such that $\mathbb{P}^K_{|\mathcal{F}_t} = G_t^K(\xi) \cdot \mathbb{P}_{|\mathcal{F}_t}$ for all $t \geq 0$. Under $\mathbb{P}^K$, the process*

$$(K \circ \xi)_t = K(t, \xi_t) - \int_0^t \frac{\partial K}{\partial s}(s, \xi_s)\mathrm{d}s, \quad t \geq 0$$

is distributed as $(\xi_t, t \geq 0)$ under $\mathbb{P}$.

(3) *If moreover we assume that for all $t \geq 0$ there exists a positive constant κ_t such that*

$$\text{for all } s \in [0, t], \quad \frac{\partial K}{\partial s}(s, \cdot) \text{ is } \kappa_t\text{-Lipschitz continuous,} \tag{3.9}$$

then $(\xi_s, s \in [0, t]) = F_t((K \circ \xi)_s, s \in [0, t])$ a.s. for some measurable $F_t : \mathcal{D}([0, t]) \mapsto \mathcal{D}([0, t])$. Then the Radon–Nikodym density of the law of $((K \circ \xi)_s, s \in [0, t])$ with respect to the law of $(\xi_s, s \in [0, t])$ is $1/G_t^K(F_t)$, i.e.

$$\mathbb{P}(K \circ \xi \in A) = \mathbb{E}\left(1_A(\xi)\frac{1}{G_t^K(F_t(\xi))}\right), \quad \forall A \in \mathcal{G}_t. \tag{3.10}$$

This result can be interpreted by saying that the law of (ξ_t) is quasi-invariant under (deterministic) non-linear transformations $\xi \mapsto K \circ \xi$. In the particular case $K(\cdot, x) \equiv K(x)$ for some time-independent K, Eqs. (3.7) and (3.9) are automatically satisfied and we have:

Corollary 3.2. *Suppose moreover that $k(s,x) = k(x)$ and $K(s,x) = K(x)$, for all $s, x \geq 0$. Then*

(1) *The process*

$$G_t^K(\xi) = \exp\left(g_0 \int_0^t \log k(\xi_s)\mathrm{d}s\right) \prod_{s \in \mathcal{J}_t^\xi} \left[k(\xi_s) \cdot \frac{g(K(\xi_s) - K(\xi_{s-}))}{g(\Delta\xi_s)}\right], \quad t \geq 0,$$

is a non-negative $(\mathcal{F}_t)$-martingale with $\mathbb{E}(G_t^K(\xi)) = 1$ and a.s. $G_t^K(\xi) > 0$.

(2) *Let $\mathbb{P}^K$ be the unique probability measure on $(\Omega, \mathcal{F})$ such that $\mathbb{P}^K_{|\mathcal{F}_t} = G_t^K(\xi) \cdot \mathbb{P}_{|\mathcal{F}_t}$ for all $t \geq 0$. Under $\mathbb{P}^K$, the process $(K(\xi_t), t \geq 0)$ is distributed as $(\xi_t, t \geq 0)$ under $\mathbb{P}$.*

(3) *The Radon–Nikodym density of the law of $(K^{-1}(\xi_s), s \in [0,t])$ with respect to the law of $(\xi_s, s \in [0,t])$ is G_t^K a.s.*

Therefore, the law of (ξ_t) is quasi-invariant under (deterministic) non-linear transformations $(\xi_t, t \geq 0) \mapsto (K(\xi_t), t \geq 0)$.

Proof of Corollary 3.1. It only remains to prove assertion (3). For all $s \in [0,t]$ let $g_s := K(s, \xi_s)$ and $R(s, \cdot)$ be the inverse of $K(s, \cdot)$ defined in Eq. (3.4). Then g satisfies the equation

$$g_u = (K \circ \xi)_u + \int_0^u \frac{\partial K}{\partial s}(s, R(s, g_s))\mathrm{d}s, \quad u \in [0,t].$$

Since the map $x \mapsto \frac{\partial K}{\partial s}(s, R(s,x))$ is Lipschitz continuous, uniformly in $s \in [0,t]$, then such an equation has a unique solution in $\mathcal{D}([0,t])$. By the classical Picard iteration procedure we obtain that $(g_s = K(s, \xi_s), s \in [0,t])$, and therefore $(\xi_s, s \in [0,t])$ itself is a measurable functional of $((K \circ \xi)_s, s \in [0,t])$. □

3.2. *Quasi-invariance of ξ under transformations of jumps*

Setting

$$H(s,x) := K(s,x), \qquad h(s,x) := k(s,x), \quad s, x \geq 0,$$

we find that Eqs. (1.1) and (1.2) are satisfied and Theorem 1.2 and Lemma 1.3 yield

Corollary 3.3. (1) *The process*

$$N_t^K(\xi) := \exp\left(g_0 \int_0^t \log k(s, 0)\mathrm{d}s\right) \prod_{s \in \mathcal{J}_t^\xi} \left[k(s, \Delta\xi_s) \cdot \frac{g(K(s, \Delta\xi_s))}{g(\Delta\xi_s)}\right], \quad t \geq 0,$$

is a non-negative $(\mathcal{F}_t)$-martingale with $\mathbb{E}(N_t^K(\xi)) = 1$ and a.s. $N_t^K(\xi) > 0$.

(2) *Let $\mathbb{P}^K$ be the unique probability measure on $(\Omega, \mathcal{F})$ such that $\mathbb{P}^K_{|\mathcal{F}_t} = N_t^K(\xi) \cdot \mathbb{P}_{|\mathcal{F}_t}$ for all $t \geq 0$. Under $\mathbb{P}^K$, the process*

$$\xi_t^K = \sum_{s \leq t} K(s, \Delta\xi_s), \quad t \geq 0,$$

is distributed as ξ under $\mathbb{P}$.

(3) *For all $t \geq 0$ we have*

$$\xi_t = \sum_{s \leq t} R(s, \Delta\xi^K_s) = \left(\xi^K\right)^R_t, \quad t \geq 0.$$

Then the Radon–Nikodym density of the law of $(\xi^K_s, s \in [0,t])$ with respect to the law of $(\xi_s, s \in [0,t])$ is $1/N^K_t(\xi^R)$, i.e.

$$\mathbb{P}(\xi^K \in A) = \mathbb{E}\left(1_A(\xi)\frac{1}{N^K_t(\xi^R)}\right), \quad \forall A \in \mathcal{G}_t. \tag{3.11}$$

This result can be interpreted by saying that the law of (ξ_t) is quasi-invariant under a non-linear (deterministic) transformation of the jumps of ξ: $(\Delta\xi_t, t \geq 0) \mapsto (K(t, \Delta\xi_t), t \geq 0)$.

Remark 3.4. In Corollary 3.3, if $k(s,x) = k(x)$ (and therefore $K(s,x) = K(x)$), then ξ^K is a subordinator with Lévy measure ν^K equal to the image measure of ν under K; in this case, the local equivalence result of Corollary 3.3 is a particular case of Sato [10, p. 217–218].

3.3. Quasi-invariance properties of the gamma process

We now write the results of Corollaries 3.1–3.3 for the special case of the gamma process (γ_t). We assume Eq. (3.7). Here

$$g(x) = \frac{e^{-x}}{x}, \quad x > 0, \qquad g_0 = 1, \qquad \Psi(\lambda) = \log(1+\lambda).$$

Corollary 3.5. *Assume Eq. (3.7). We set for all $t \geq 0$*

$$\begin{aligned} Y^K_t(\gamma) := \exp&\left(\gamma_t - K(t,\gamma_t) + \int_0^t \left(\frac{\partial K}{\partial s} + \log k\right)(s,\gamma_s)\mathrm{d}s\right) \\ &\times \prod_{s \in \mathcal{J}^\gamma_t}\left[\frac{k(s,\gamma_s)\cdot\Delta\gamma_s}{K(s,\gamma_s) - K(s,\gamma_{s-})}\right]. \end{aligned} \tag{3.12}$$

(1) *Then $(Y^K_t(\gamma))$ is a martingale with $\mathbb{E}(Y^K_t(\gamma)) = 1$ and a.s. $Y^K_t(\gamma) > 0$.*
(2) *Let $\mathbb{P}^K$ be the unique probability measure on $(\Omega, \mathcal{F})$ such that $\mathbb{P}^K_{|\mathcal{F}_t} = Y^K_t(\gamma)\cdot\mathbb{P}_{|\mathcal{F}_t}$ for all $t \geq 0$. Under $\mathbb{P}^K$, the process*

$$(K \circ \gamma)_t := K(t,\gamma_t) - \int_0^t \frac{\partial K}{\partial s}(s,\gamma_s)\mathrm{d}s, \quad t \geq 0$$

is distributed as $(\gamma_t, t \geq 0)$ under $\mathbb{P}$.
(3) *Under the additional assumption Eq. (3.9), $(\gamma_s, s \in [0,t]) = F_t((K\circ\gamma)_s, s \in [0,t])$ a.s. for some measurable $F_t : \mathcal{D}([0,t]) \mapsto \mathcal{D}([0,t])$. Then the Radon–Nikodym density of the law of $((K\circ\gamma)_s, s \in [0,t])$ with respect to the law of $(\gamma_s, s \in [0,t])$ is $1/Y^K_t(F_t)$, i.e.*

$$\mathbb{P}(K\circ\gamma \in A) = \mathbb{E}\left(1_A(\gamma)\frac{1}{Y^K_t(F_t(\gamma))}\right), \quad \forall A \in \mathcal{G}_t. \tag{3.13}$$

Corollary 3.6. *Suppose that $k(s,x) = k(x)$ and $K(s,x) = K(x)$, for all $s, x \geq 0$. Then:*

(1) *The process*

$$Y_t^K(\gamma) := \exp\left(\gamma_t - K(\gamma_t) + \int_0^t \log k(\gamma_s)\mathrm{d}s\right) \prod_{s\in\mathcal{J}_t^\gamma}\left[\frac{k(\gamma_s)\cdot\Delta\gamma_s}{K(\gamma_s)-K(\gamma_{s-})}\right],\quad t\geq 0 \tag{3.14}$$

is a martingale with $\mathbb{E}(Y_t^K(\gamma)) = 1$ *and a.s.* $Y_t^K(\gamma) > 0$.

(2) *Let* $\mathbb{P}^K$ *be the unique probability measure on* $(\Omega,\mathcal{F})$ *such that* $\mathbb{P}^K_{|\mathcal{F}_t} = Y_t^K(\gamma)\cdot\mathbb{P}_{|\mathcal{F}_t}$ *for all* $t\geq 0$. *Under* $\mathbb{P}^K$, *the process* $(K(\gamma_t), t\geq 0)$ *is distributed as* $(\gamma_t, t\geq 0)$ *under* $\mathbb{P}$.

(3) *The Radon–Nikodym density of the law of* $(K^{-1}(\gamma_s), s\in[0,t])$ *with respect to the law of* $(\gamma_s, s\in[0,t])$ *is* Y_t^K.

Corollary 3.7. (1) *The process*

$$Z_t^K(\gamma) := \exp\left(\gamma_t - \sum_{s\leq t} K(s,\Delta\gamma_s) + \int_0^t \log k(s,0)\mathrm{d}s\right) \times \prod_{s\in\mathcal{J}_t^\gamma}\left[k(s,\Delta\gamma_s)\cdot\frac{\Delta\gamma_s}{K(s,\Delta\gamma_s)}\right],$$

$t\geq 0$, *is a non-negative* $(\mathcal{F}_t)$*-martingale with* $\mathbb{E}(Z_t^K(\gamma)) = 1$ *and a.s.* $Z_t^K(\gamma) > 0$.

(2) *Let* $\mathbb{P}^K$ *be the unique probability measure on* $(\Omega,\mathcal{F})$ *such that* $\mathbb{P}^K_{|\mathcal{F}_t} = Z_t^K(\gamma)\cdot\mathbb{P}_{|\mathcal{F}_t}$ *for all* $t\geq 0$. *Under* $\mathbb{P}^K$, *the process*

$$\gamma_t^K = \sum_{s\leq t} K(s,\Delta\gamma_s),\quad t\geq 0,$$

is distributed as $(\gamma_t, t\geq 0)$ *under* $\mathbb{P}$.

(3) *For all* $t\geq 0$ *we have*

$$\gamma_t = \sum_{s\leq t} R(s,\Delta\gamma_s^K) = \left(\gamma^K\right)_t^R,\quad t\geq 0.$$

Then the Radon–Nikodym density of the law of $(\gamma_s^K, s\in[0,t])$ *with respect to the law of* $(\gamma_s, s\in[0,t])$ *is* $1/Z_t^K(\gamma^R)$, *i.e.*

$$\mathbb{P}(\gamma^K\in A) = \mathbb{E}\left(1_A(\gamma)\frac{1}{Z_t^K(\gamma^R)}\right),\quad \forall A\in\mathcal{G}_t. \tag{3.15}$$

Remark 3.8. If $K(s,x) = a_s x$ for a measurable function $a:\mathbb{R}_+\mapsto\mathbb{R}_+$ with a and $1/a$ bounded, then the result of Corollary 3.1 has been obtained in [12]; our main result, Theorem 1.2, yields a much more general statement, which holds for a predictable (although in the general case the Radon–Nikodym density is not always explicit). In Corollary 3.7, if $k(s,x) = k(x)$ (and therefore $K(s,x) = K(x)$), then γ^K is a subordinator with Lévy measure ν^K equal to the image measure of $\frac{\mathrm{e}^{-x}}{x}1_{(x>0)}\mathrm{d}x$ under K; in this case, the local equivalence result of Corollary 3.3 is a particular case of Sato [10, p. 217–218].

4. Quasi-invariance properties of the Dirichlet process

We fix $T > 0$ and we denote by $(D_t^{(T)} : t \in [0, T])$ the Dirichlet process over the time interval $[0, T]$, i.e. $D_t^{(T)} := \gamma_t/\gamma_T$, $t \in [0, T]$, where (γ_t) is a gamma process. Since T is fixed we omit the superscript (T). We consider measurable functions K and k satisfying Eqs. (3.1)–(3.3).

4.1. Quasi-invariance of D under composition with a diffeomorphism

We want to give a martingale proof of a relation originally obtained by von Renesse and Sturm in [7]. In this subsection we also assume that $k(\cdot, y) = k(y)$ is time independent for all $y \in \mathbb{R}_+$ and satisfies

$$\int_0^1 k(y)\mathrm{d}y = 1,$$

so that $K(s, x) = K(x) = \int_0^x k(y)\mathrm{d}y$, $x \in [0, 1]$, also satisfies

$$K(0) = 0, \qquad K(1) = 1.$$

Notice that $K(\cdot) : [0, 1] \mapsto [0, 1]$ is necessarily bijective. We set for $t < T$

$$L_t^{K,T} := \left(\frac{1 - K(D_t)}{1 - D_t}\right)^{T-t-1} \exp\left(\int_0^t \log k(D_s)\mathrm{d}s\right) \prod_{s \in \mathcal{J}_t^D} \left[\frac{k(D_s) \cdot \Delta D_s}{K(D_s) - K(D_{s-})}\right],$$

$$L_T^{K,T} := \frac{1}{k(1)} \exp\left(\int_0^T \log k(D_s)\mathrm{d}s\right) \prod_{s \in \mathcal{J}_T^D} \left[\frac{k(D_s) \cdot \Delta D_s}{K(D_s) - K(D_{s-})}\right].$$

Theorem 4.1. (1) $(L_t^{K,T}, t \in [0, T])$ *is a martingale with respect to the natural filtration of* D, *such that* $\mathbb{E}(L_t^{K,T}) = 1$ *and a.s.* $L_t^{K,T} > 0$, *for all* $t \in [0, T]$.

(2) *Under* $\mathbb{P}^{K,T} := L_T^{K,T} \cdot \mathbb{P}$, *the process* $(K(D_t), t \in [0, T])$ *has the same law as* $(D_t, t \in [0, T])$ *under* $\mathbb{P}$.

(3) *The Radon–Nikodym density of the law of* $(K^{-1}(D_t), t \in [0, T])$ *with respect to the law of* $(D_t, t \in [0, T])$ *is* $L_T^{K,T}$.

This theorem expresses the quasi-invariance of the law of D under non-linear transformations $(D_s, s \in [0, T]) \mapsto (K(D_s), s \in [0, T])$.

Remark 4.2. In [7], von Renesse and Sturm prove the result of Theorem 4.1 using explicit computations on the finite dimensional distributions of D. Our proof clarifies the structure of the Radon–Nikodym density and shows its links with more general quasi-invariance phenomena.

Proof of Theorem 4.1. Let first $t < T$. By the Markov property, for all bounded Borel $\Phi : \mathcal{D}([0, t]) \mapsto \mathbb{R}_+$

$$\begin{aligned}\mathbb{E}\left(\Phi(D_s, s \le t)\right) &= \mathbb{E}\left(\Phi(\gamma_s, s \le t)1_{(\gamma_t < 1)} \frac{p_{T-t}(1 - \gamma_t)}{p_T(1)}\right) \\ &= \mathbb{E}\left(\Phi(\gamma_s, s \le t)1_{(\gamma_t < 1)}(1 - \gamma_t)^{T-t-1}\mathrm{e}^{\gamma_t}\right) \frac{\Gamma(T)}{\Gamma(T - t)}.\end{aligned}$$

Let us consider the process (Y_t^K) as defined in Eq. (3.12). In the time-independent case we have the simpler expression

$$Y_t^K := \exp\left(\gamma_t - K(\gamma_t) + \int_0^t \log k(\gamma_s)\mathrm{d}s\right) \prod_{s \in \mathcal{J}_t^\gamma} \left[\frac{k(\gamma_s) \cdot \Delta\gamma_s}{K(\gamma_s) - K(\gamma_{s-})}\right].$$

Notice that $K(\cdot)$ is strictly increasing and $K(1) = 1$, so that $K(\gamma_t) < 1$ iff $\gamma_t < 1$. Then, for all bounded Borel $\Phi : \mathcal{D}([0,t]) \mapsto \mathbb{R}_+$, $t < T$, by Corollary 3.5

$$\begin{aligned}\mathbb{E}\left(\Phi(K(D.))L_t^{K,T}\right) &= \frac{\Gamma(T)}{\Gamma(T-t)}\mathbb{E}\left(1_{(\gamma_t<1)}\,(1-K(\gamma_t))^{T-t-1}\,\Phi(K(\gamma.))\mathrm{e}^{K(\gamma_t)}Y_t^K\right)\\ &= \frac{\Gamma(T)}{\Gamma(T-t)}\mathbb{E}\left(1_{(\gamma_t<1)}\,(1-\gamma_t)^{T-t-1}\,\Phi(\gamma)\mathrm{e}^{\gamma_t}\right) = \mathbb{E}(\Phi(D.)),\end{aligned}$$

and this concludes the proof for $t < T$.

We consider now the case $t = T$. For all bounded Borel $\Phi : \mathcal{D}([0,T]) \mapsto \mathbb{R}_+$ and $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+$, by Corollary 3.5

$$\mathbb{E}\left(\Phi(K(\gamma.))\varphi(K(\gamma_T))\,Y_T^K\right) = \mathbb{E}(\Phi(\gamma)\varphi(\gamma_T)). \tag{4.1}$$

We set for all $x > 0$

$$Y_T^{K,x} := \exp\left(x - K(x) + \int_0^T \log k(xD_s)\mathrm{d}s\right) \prod_{s \in \mathcal{J}_T^D} \left[\frac{k(xD_s) \cdot x\Delta D_s}{K(xD_s) - K(xD_{s-})}\right].$$

On the right hand side of Eq. (4.1) we condition on the value of γ_T, obtaining

$$\mathbb{E}\left(\Phi(\gamma.)\varphi(\gamma_T)\right) = \int_0^\infty p_T(y)\mathbb{E}\left(\Phi(yD.)\right)\varphi(y)\mathrm{d}y. \tag{4.2}$$

On the left hand side of Eq. (4.1), conditioning on the value of γ_T, we obtain

$$\mathbb{E}\left(\Phi(K(\gamma.))\,\varphi(K(\gamma_T))Y_T^K\right) = \int_0^\infty p_T(x)\mathbb{E}\left(\Phi(K(xD.))Y_T^{K,x}\right)\varphi(K(x))\mathrm{d}x. \tag{4.3}$$

In order to compare Eqs. (4.2) and (4.3), we use the change of variable $x = K(y)$. To this end, we denote by $C : \mathbb{R}_+ \mapsto \mathbb{R}_+$ the inverse of $K(\cdot)$, i.e. we suppose that $K(C(x)) = x$ for all $x \geq 0$. Then we have

$$\begin{aligned}\mathbb{E}\left(\Phi(K(\gamma.))\,\varphi(K(\gamma_T))Y_T^K\right) &= \int_0^\infty p_T(x)\mathbb{E}\left(\Phi(K(xD.))Y_T^{K,x}\right)\varphi(K(x))\mathrm{d}x\\ &= \int_0^\infty p_T(C(y))\mathbb{E}\left(\Phi(K(C(y)D.))Y_T^{K,C(y)}\right)\varphi(y)C'(y)\mathrm{d}y.\end{aligned}$$

Since this is true for any bounded measurable $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+$, we obtain for all $y > 0$

$$\frac{p_T(C(y))C'(y)}{p_T(y)}\mathbb{E}\left(\Phi(K(C(y)D.))\,Y_T^{K,C(y)}\right) = \mathbb{E}(\Phi(yD.)).$$

For $y = 1$, since $K(1) = 1 = C(1)$ and $C'(1) = 1/k(1)$, we obtain the desired result

$$\mathbb{E}\left(\Phi(K(D.))L_T^{K,T}\right) = \mathbb{E}(\Phi(D.)).$$

In particular, applying this formula to the inverse R of K we obtain the last assertion:

$$\mathbb{E}\left(\Phi(K(D.))\right) = \mathbb{E}\left(\Phi(D.)L_T^{R,T}\right). \quad \square$$

4.2. Quasi-invariance of D under transformation of the jumps

Again, we consider the Dirichlet process $(D_t^{(T)}, t \in [0, T])$, and we drop the superscript (T), since T is fixed. We set

$$\Delta D_s := D_s - D_{s-}, \quad \mathcal{D}_t^K := \frac{\sum_{s\le t} K(s, \Delta D_s)}{\sum_{s\le T} K(s, \Delta D_s)} = \frac{D_t^K}{D_T^K}, \quad t \in [0, T].$$

Theorem 4.3. *The laws of $(\mathcal{D}_t^K, t \in [0, T])$ and $(D_t, t \in [0, T])$ are equivalent.*

Remark 4.4. Handa [5] proves Theorem 4.3 in the particular case $K(s, x) = c(s)x$, where $c : [0, T] \mapsto \mathbb{R}_+$ is measurable. An expression for the Radon–Nikodym density might be obtained from the proof of Theorem 4.3: see Eq. (4.7).

Proof. Since $(D_t, t \in [0, T])$ is a gamma bridge, then the law of $(\mathcal{D}_t^K, t \in [0, T])$ coincides with the law of $(\gamma_t^K/\gamma_T^K, t \in [0, T])$ under the conditioning $\{\gamma_T = 1\}$.

Consider now R, defined in Eq. (3.4), and recall that

$$\left(\gamma^K\right)^R = \gamma.$$

By Corollary 3.7, for all $\Phi : \mathcal{D}([0, T]) \mapsto \mathbb{R}$ bounded and Borel

$$\mathbb{E}\left(\Phi\left(\frac{\gamma_s^K}{\gamma_T^K}, s \le T\right)\varphi(\gamma_T)Z_T^K\right) = \mathbb{E}\left(\Phi\left(\frac{\gamma_s}{\gamma_T}, s \le T\right)\varphi(\gamma_T^R)\right). \tag{4.4}$$

Note that

$$\gamma_T^R = \sum_{s\le T} R(s, \Delta\gamma_s) = \sum_{s\le T} R\left(s, \gamma_T \cdot \Delta D_s\right) =: \psi_D(\gamma_T),$$

where $D_t := \gamma_t/\gamma_T$, $t \in [0, T]$, is independent of γ_T and

$$\psi_D(x) := \sum_{s\le T} R\left(s, x \cdot \Delta D_s\right), \quad x \ge 0.$$

Note that $\psi_D : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is C^1 and by dominated convergence

$$\psi_D'(x) = \sum_{s\le T} \Delta D_s \cdot \frac{\partial R}{\partial x}\left(s, x \cdot \Delta D_s\right) \ge \kappa^{-1} > 0, \quad \forall x \ge 0,$$

since $\Delta D_s \ge 0$ and $\sum_{s\le T} \Delta D_s = 1$. Also by dominated convergence, ψ_D' is continuous. Therefore $\psi_D : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is invertible, with C^1 inverse $\zeta_D := \psi_D^{-1}$. In the sequel, we may write ζ_D^K for ζ_D, in order to stress that it also depends on K. Then, for all $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}$ bounded and Borel, we obtain by Eq. (4.4)

$$\begin{aligned}
\mathbb{E}\left(\Phi\left(\frac{\gamma_s^K}{\gamma_T^K}, s \le T\right)\varphi(\gamma_T)Z_T^K\right) &= \mathbb{E}\left(\Phi\left(D_s, s \le T\right)\varphi(\psi_D(\gamma_T))\right) \\
&= \mathbb{E}\left(\Phi\left(D_s, s \le T\right)\int_0^\infty p_T(y)\varphi(\psi_D(y))\mathrm{d}y\right) \\
&= \int_0^\infty \varphi(x)\mathbb{E}\left(\Phi\left(D_s, s \le T\right)p_T(\zeta_D(x))\zeta_D'(x)\right)\mathrm{d}x, \qquad (4.5)
\end{aligned}$$

after performing the change of variable $x = \psi_D(y)$. Now, setting for all $t \in [0, T]$

$$\mathcal{D}_t^{K,x} := \frac{\sum_{s \le t} K(s, x \cdot \Delta D_s)}{\sum_{s \le T} K(s, x \cdot \Delta D_s)},$$

$$U_T^{K,x} := \exp\left(x - \sum_{s \le T} K(s, x\Delta D_s) + \int_0^T \log k(s,0)\mathrm{d}s\right)\prod_{s \in \mathcal{J}_T^D}\left[\frac{k(s, x\Delta D_s)\cdot x\Delta D_s}{K(s, x\Delta D_s)}\right],$$

we obtain

$$\mathbb{E}\left(\Phi\left(\frac{\gamma_s^K}{\gamma_T^K}, s \le T\right)\varphi(\gamma_T)Z_T^K\right) = \int_0^\infty \varphi(x)\mathbb{E}\left(\Phi\left(\mathcal{D}_s^{K,x}, s \le T\right)\cdot U_T^{K,x}\right)p_T(x)\mathrm{d}x. \quad (4.6)$$

Since $\mathcal{D}^{K,1} = \mathcal{D}^K$, setting

$$U_T^K(D) := U_T^{K,1} = \exp\left(1 - D_T^K + \int_0^T \log k(s,0)\mathrm{d}s\right)\prod_{s \in \mathcal{J}_T^D}\left[\frac{k(s, \Delta D_s)\cdot \Delta D_s}{K(s, \Delta D_s)}\right],$$

we obtain by Eqs. (4.5) and (4.6) for $x = 1$

$$\mathbb{E}\left(\Phi\left(\mathcal{D}_s^K, s \le T\right)\cdot U_T^K\right) = \mathbb{E}\left(\Phi\left(D_s, s \le T\right)\frac{p_T(\zeta_D^K(1))}{p_T(1)}(\zeta_D^K)'(1)\right). \quad \square \qquad (4.7)$$

5. Stochastic differential equations driven by ($\mathcal{L}$)-subordinators

In this section we give an application of the previous results to stochastic differential equations driven by a ($\mathcal{L}$)-subordinator ξ. See [2] for a survey of SDEs driven by Lévy processes.

We consider the SDE

$$\mathrm{d}x_t = m(t, x_{t-})\mathrm{d}\xi_t, \quad x_0 = 0, \qquad (5.1)$$

where

(1) $m : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto]0, +\infty[$ is measurable;
(2) m and $1/m$ are bounded;
(3) $\mathbb{R}_+ \ni a \mapsto m(s, a)$ is Lipschitz, uniformly in $s \ge 0$.

Then we have

Theorem 5.1. *There exists a pathwise-unique solution of Eq.* (5.1) *and the law of* (x, ξ) *under* $\mathbb{P}$ *coincides with the law of* (ξ, ξ^H) *under* $\mathbb{P}^H$*, where*

$$H(s, x) := \frac{x}{m(s, \xi_{s-})}, \quad s \geq 0,\ x \geq 0. \tag{5.2}$$

Proof. For any interval $J \subset \mathbb{R}$, denote by $\mathcal{I}(J)$ the set of all bounded increasing functions $\omega : J \mapsto \mathbb{R}_+$.

We prove now that there exists a.s. a unique pathwise solution of Eq. (5.1). Let L be the Lipschitz constant of $m(s, \cdot)$, which exists by the uniformity assumption (3) above. We define the sequence of random times

$$T_0 := 0, \qquad T_{i+1} := \sup\left\{t \in]T_i, T] : \xi_{t-} - \xi_{T_i} < \frac{1}{2L}\right\}, \quad \sup\emptyset := T.$$

If $T_i < T$, then $\xi_T \geq \xi_{T_i} \geq \frac{i}{2L}$, so that the cardinality N of $\{i : T_{i-1} < T_i\}$ is bounded above by $2L\xi_T < \infty$ a.s.

Now we set $x_0 := x$. Suppose that we have proven existence and uniqueness of the solution x of Eq. (5.1) over $[0, T_{i-1}]$ for $i \in \{1, \ldots, N\}$. We define the map $\Lambda_{T_{i-1},T_i} : \mathcal{I}([T_{i-1}, T_i[) \mapsto \mathcal{I}([T_{i-1}, T_i[)$

$$\Lambda_{T_{i-1},T_i}(\omega)(t) := x_{T_{i-1}} + \int_{[T_{i-1},t]} m(s, \omega_{s-})\mathrm{d}\xi_s, \quad t \in [T_{i-1}, T_i[.$$

Then Λ_{T_{i-1},T_i} is a contraction in $\mathcal{I}([T_{i-1}, T_i[)$ with respect to the sup-metric:

$$|\Lambda_{T_{i-1},T_i}(\omega)(t) - \Lambda_{T_{i-1},T_i}(\omega')(t)| \leq \sup_{s\in[T_{i-1},T_i[} |m(s, \omega_s) - m(s, \omega'_s)| \cdot \left(\xi_{T_i-} - \xi_{T_{i-1}}\right),$$

for all $t \in [T_{i-1}, T_i[$, which yields

$$\sup_{[T_{i-1},T_i[} |\Lambda_{T_{i-1},T_i}(\omega) - \Lambda_{T_{i-1},T_i}(\omega')| \leq L \cdot \frac{1}{2L} \sup_{[T_{i-1},T_i[} |\omega - \omega'| = \frac{1}{2} \sup_{[T_{i-1},T_i[} |\omega - \omega'|.$$

Then the solution x of Eq. (5.1) over $[0, T_i[$ coincides over $[T_{i-1}, T_i[$ with the unique fixed point x of Λ_{T_{i-1},T_i}. Now, for $t = T_i$ we have necessarily by Eq. (5.1)

$$x_{T_i} = x_{T_i-} + m(T_i, x_{T_i-})(\xi_{T_i} - \xi_{T_i-}).$$

By recurrence, we obtain existence and uniqueness of a pathwise solution of Eq. (5.1) over $[0, T]$. Moreover, there exists a measurable map $W_T : \mathcal{I}([0, T]) \mapsto \mathcal{I}([0, T])$, such that $x = W_T(\xi_{|[0,T]})$.

Let us define H as in Eq. (5.2), and set ξ^H as in Eq. (1.4)

$$\xi_t^H := \sum_{s\leq t} H(s, \Delta\xi_s) = \int_0^t \frac{1}{m(s, \xi_{s-})}\mathrm{d}\xi_s$$

Note that

$$\mathrm{d}\xi_t^H = \frac{1}{m(t, \xi_{t-})}\mathrm{d}\xi_t \Longrightarrow \mathrm{d}\xi_t = m(t, \xi_{t-})\mathrm{d}\xi_t^H.$$

Then, $\xi_{|[0,T]} = W_T(\xi^H_{|[0,T]})$ for any $T > 0$. On the other hand, by Theorem 1.2, ξ^H under $\mathbb{P}^H$ has the same law as ξ under $\mathbb{P}$, and this concludes the proof. $\square$

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